MST121 CB C

The Open University

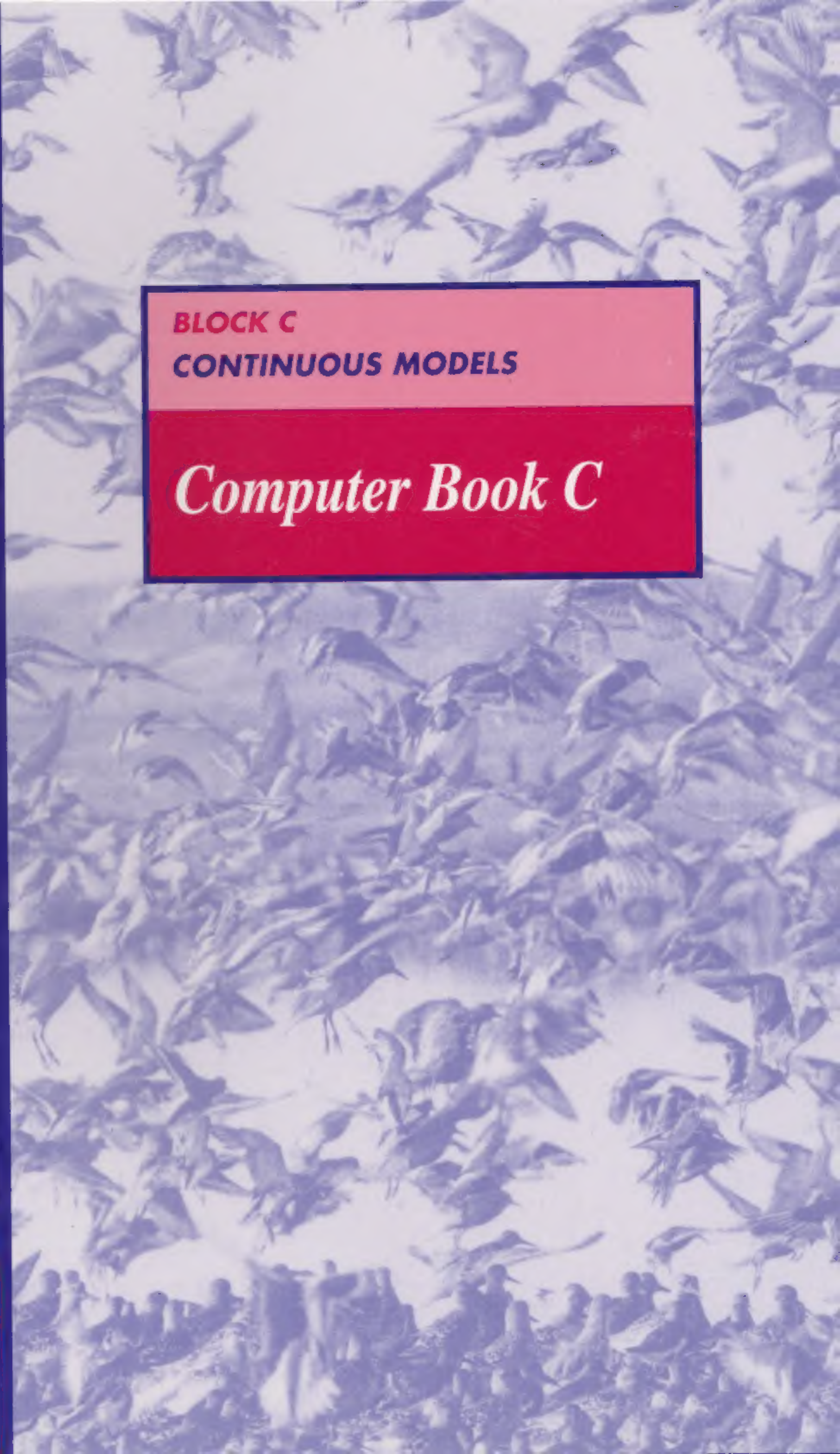
A first level interdisciplinary course

Using Mathematics

BLOCK C

CONTINUOUS MODELS

Computer Book C



COMPUTER BOOK

C

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Prepared by the course team

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About this course

This computer book forms part of the course MST121 *Using Mathematics*. This course and the courses MU120 *Open Mathematics* and MS221 *Exploring Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MST121 uses the software program Mathcad (MathSoft, Inc.) and other software to investigate mathematical and statistical concepts and as a tool in problem solving. This software is provided as part of the course.

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2.1

Contents

Guidance notes

This computer book contains those sections of the chapters in Block C which require you to use Mathcad. Each of these chapters contains instructions as to when you should first refer to particular material in this computer book, so you are advised not to work on the activities here until you have reached the appropriate points in the chapters.

In order to use this computer book, you will need the following Mathcad files.

Chapter C1

121C1-01 Differentiation

Chapter C2

121C2-01 Integration as a limit of summations (Optional)

Chapter C3

121C3-01 Direction fields and solution curves
121C3-02 Direction fields, solution curves and Euler's method

Instructions for installing these files onto your computer's hard disk, and for opening them, are given in Chapter A0.

The computer activities for Chapters C1 and C2 require you also to work with Mathcad documents which you have created yourself.

Activities based on software vary both in nature and in length. Sometimes the instructions for an activity appear only in the computer book; in other cases, instructions are given in the computer book and on screen. Feedback on an activity is sometimes provided on screen and is sometimes given in the computer book.

For advice on how each computer session fits into the suggested study patterns, refer to the Study guides in the chapters.

Chapter C1, Section 5
Optimisation with the computer

The Mathcad activities for Chapter C1 have only one prepared file associated with them. This file explains how to differentiate with the computer, which is the topic of Subsection 5.1. To solve optimisation problems, in Subsection 5.2, you will be given guidance on how to create your own Mathcad documents.

By the end of this section, you should be able to use Mathcad to carry out the process of differentiation and to solve optimisation problems, and also be a more skilled and independent Mathcad user.

5.1 Finding derivatives

In Mathcad, derivatives are found using the $\frac{d}{dx}$ operator. This operator can be used in two different ways, depending on whether you simply wish to find the numerical value of the derivative of a function f at a particular point, or whether you seek a general formula for the derivative of f.

For example, the derivative of the function $f(x) = \frac{1}{5}x^2$ can be described by the formula $f'(x) = \frac{2}{5}x$, which holds for all real values of x. Mathcad can obtain this expression for the derivative, using the $\frac{d}{dx}$ operator *symbolically*, as indicated in Activity 5.2. The value of the derivative $f'(x) = \frac{2}{5}x$ at the particular point $x = 3$ can then be found as $f'(3) = \frac{2}{5} \times 3 = 1.2$. However, it is also possible to obtain the particular value $f'(3) = 1.2$ *without* first finding the symbolic expression $\frac{2}{5}x$ for the derivative. Mathcad can achieve this by using the $\frac{d}{dx}$ operator *numerically*, as you will see in Activity 5.1.

Essentially, this involves a direct application of the definition of derivative, as given by equation (1.3) in Chapter C1 (with, in this case, $x = 3$ and $f(x) = \frac{1}{5}x^2$).

Activity 5.1 Evaluating a derivative numerically

Open Mathcad file 121C1-01. Page 1 introduces the document. Work through page 2, and then carry out Task 1 on page 3.

Don't forget to make your own working copy of the file.

Solutions are given on page 31.

Comment

- It is a good idea to adopt a systematic approach when finding the two points at which the derivative of $f(x) = x^3 - x^2 - 3x + 4$ is zero. For example, try a value that looks reasonable from the graph, say $x = 1.5$, which gives the derivative value 0.75. This value is positive (as is also apparent visually from the slope of the corresponding tangent plotted in blue on the graph). Since the value of x sought here is a minimum of the graph of f, the positive derivative indicates that $x = 1.5$ is too large an estimate. Hence try further values for x, each 0.1 less than the previous one, until (for $x = 1.3$) the value given for the derivative is negative. This indicates that a value for x at which $f'(x) = 0$ is somewhere between the last two x-values tried, 1.3 and 1.4. If a greater degree of accuracy is required, then this process can be repeated by trying further values of x between these last two, and continuing until the desired level of accuracy is obtained.
- The main aim of this activity was to illustrate the Mathcad facility for evaluating derivatives numerically at given points, though it should also have reinforced the link between the value of the derivative at

various points and features of the graph of the function. In order to find where the function $f(x) = x^3 - x^2 - 3x + 4$ has derivative zero, it is, in practice, quicker to find the formula for the derivative $f'(x)$, set it equal to zero, and solve the resulting equation, $3x^2 - 2x - 3 = 0$.

Remember that Mathcad notes are *optional*.

Mathcad notes

- ◇ When evaluating a derivative in this way (numerically), you must define earlier in the document the point at which the derivative is to be found, e.g. $x := 3$. You can then either enter an expression directly into the derivative operator, e.g. $x^2/5$, or enter the name of a function, e.g. $f(x)$, which has been defined previously. (These two approaches are shown on pages 2 and 3 of the document, respectively.)
- ◇ When you type the equals sign = to evaluate an expression numerically, it doesn't matter where on the expression the blue bar cursor is, or if all or part of the expression is enclosed by the blue selection box. All that matters is that the expression is complete, with every placeholder filled in.
- ◇ When an expression involving the $\frac{d}{dx}$ operator is evaluated numerically, Mathcad uses a numerical method to obtain an approximation to the exact value of the derivative at a point, which is usually accurate to 7 or 8 significant figures. Very occasionally the method fails, in which case Mathcad displays the error message 'not converging'.

Once the symbolic processor is loaded, you can use Mathcad to differentiate a function symbolically and hence obtain an algebraic expression for the derivative. This replicates what you might do by hand, but Mathcad can be used to differentiate functions that would be rather complicated to do by hand.

Activity 5.2 Differentiating symbolically

You should still be working with Mathcad file 121C1-01.

Work through page 4 of the document to load the symbolic processor and set up how the results are displayed. Then follow the instructions to differentiate the function $x^2/5$ symbolically.

Carry out Task 2 on page 5 of the document.

Solutions are given on page 31.

Comment

- ◇ Each of parts (b)–(h) in Task 2 concerns a function that you were asked to differentiate in the main text. In the solutions, where the expression obtained using Mathcad differs in appearance from that obtained by hand, both expressions are given. If you cannot see immediately why such a pair of expressions are equivalent, it is a good idea to copy down one expression and use algebra to verify that it can be rearranged to give the other.
- ◇ Sometimes the form of solution obtained by Mathcad can be 'improved' by simplifying it, as follows. After differentiating symbolically, leave the derivative selected (enclosed in the blue selection box) and choose **Simplify** from the **Symbolic** menu. In many cases **Factor Expression**, also from the **Symbolic** menu, can be used to simplify an expression. However, these options should be used with caution, as what Mathcad regards as 'simpler' may not necessarily seem so to a human observer!

These commands are introduced in Chapter A0, file 121A0-05.

- Make sure that the variables entered in the two placeholders of the derivative operator match. For example, if you mistakenly try to evaluate the expression

  $$\frac{d}{dt}\cos(4\cdot x),$$

  then you will obtain the answer 0.

- Brackets can be crucial when entering an expression into the right-hand placeholder of the derivative operator. For example, if the expression $x^3 - x^2 - 3x + 4$ is entered without being enclosed in brackets, Mathcad will differentiate the x^3 term but not the others.

  You may find that Mathcad will not allow you to enter into the placeholder a quotient whose numerator is a sum of terms (it depends on how you enter the quotient). This difficulty can be avoided by enclosing the quotient in brackets.

  An example of such a quotient is

  $$\frac{\sin(t)-t^2}{e^t}.$$

  To avoid such problems, it is good practice always to include outer brackets around expressions that you enter into the right-hand placeholder of the $\frac{d}{dx}$ operator. This also helps to make the expression to be differentiated clear on the screen.

Mathcad notes

- When Mathcad carries out *symbolic* calculations, it does not refer to any variable or function definitions that occur in the document. For example, symbolic evaluation of the expression

  $$\frac{d}{dx}(a\cdot x^2) \quad \text{yields} \quad 2\cdot a\cdot x,$$

  whether or not a has been defined. Similarly, symbolic evaluation of the expression

  $$\frac{d}{dx}f(x) \quad \text{yields} \quad \frac{d}{dx}f(x),$$

  whether or not the function f has been defined. In this case, Mathcad is unable to evaluate the expression and simply reproduces it.

  Both the above expressions can be evaluated *numerically* in the usual way, provided that the variables x and a, and the function f, have been defined previously in the document.

- The expressions e^x and $\exp(x)$ are equivalent in Mathcad, but the latter form is always used in the output of symbolic calculations, irrespective of the form used for input.

- Mathcad carries out a symbolic calculation only when you select an expression and choose a command from the **Symbolic** menu or use the corresponding keyboard shortcut. The result of a symbolic calculation is *not* updated if you change the original expression (unlike the result of a numerical calculation).

- You may have noticed the Mathcad command **Differentiate on Variable** in the **Symbolic** menu. This provides an alternative approach to finding a formula for the derivative of a function, which is described in *A Guide to Mathcad*. However, it is usually better to use the $\frac{d}{dx}$ operator and **Evaluate Symbolically**, as described in the current activity, because this results in clearer Mathcad documents.

Now close file 121C1-01.

5.2 Optimisation

See Chapter C1, Subsection 2.3.

Optimisation involves finding the greatest or least value taken by a function on an interval. In the main text you saw how to apply the Optimisation Procedure by hand, but all the calculations required can also be performed using Mathcad. In some cases, Mathcad can be applied to check calculations already done by hand, while in other cases, calculations can be carried out that are too complicated to do by hand.

In this subsection you will solve two optimisation problems, the first a minimisation problem and the second a maximisation problem.

The orienteer's problem

See Computer Book A, Chapter A3, Subsection 5.5.

The orienteer's problem was described in Computer Book A. The orienteer starts at a point in a forest and needs to reach a final point on a path. The problem is to find where the orienteer should aim to join the path, where the joining point is x km from a fixed point O on the path, in order to minimise the time taken overall. For the particular data given, this involves finding the value of x that gives the least value of the journey time (in hours)

$$f(x) = 0.125\sqrt{1+x^2} + 0.0625(2-x) \quad (x \text{ in } [0,2]).$$

The graph of $y = f(x)$ is shown in Figure 5.1.

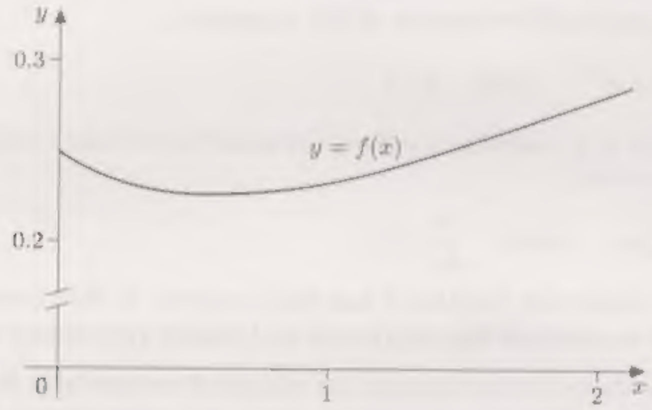


Figure 5.1 Graph of the function for the orienteer's problem

In Computer Book A, you found an approximate solution to this problem using Mathcad, by zooming in on the graph and using the crosshair. You are now in a position to solve the problem more accurately, using differentiation.

In Activity 5.3 you are asked to use the Optimisation Procedure to solve the orienteer's problem. In place of a prepared Mathcad file for this activity, you are guided through creating your own Mathcad document. This includes a requirement to enter some explanatory text, to make the document more comprehensible.

The instructions for this activity may look lengthy, but many of them describe general Mathcad techniques that can save you time and effort, now and in the future. Most of the instructions are given initially in a mouse/click way, while details of keyboard alternatives are provided in the first Mathcad note for the activity.

Activity 5.3 Minimising the orienteer's journey time

The following instructions guide you through creating a Mathcad document to carry out the Optimisation Procedure, in order to solve the orienteer's problem. Part (b) describes how to enter a title in the document. You should similarly enter any other text that you think is appropriate. For example, you could include a line of text before each step of the Optimisation Procedure, to explain what you are doing.

(a) Begin by creating a new blank document, as follows. With Mathcad running on your computer, select the **File** menu and choose **New**. (If you have just started Mathcad running, then there is no need to do this, as it automatically starts with a new document.)

A Guide to Mathcad contains detailed information on creating and editing your own documents.

(b) Enter a title at the top of your document. To do this, click to position the red cross cursor in an appropriate place, and choose **Create Text Region** from the **Text** menu. Then type a suitable title, for example, `Optimisation - the orienteer's problem`, in the text box. To finish, click anywhere outside the text box. If you need to edit the text later, simply click on it.

Any text that you enter can later be moved, or deleted, as you wish.

(c) Load the symbolic processor by choosing **Load symbolic processor** from the **Symbolic** menu. (You will be using both symbolic differentiation and symbolic solution of an equation in the optimisation process.)

If you have already loaded the symbolic processor during this Mathcad session, then you will not need to do so now

It is also helpful to use the Mathcad facility for providing comments, by selecting **Derivation Format...** from the **Symbolic** menu, clicking in the check box to 'Show derivation comments' and selecting the option button to show the derivation steps 'vertically, inserting lines', then clicking 'OK'.

The derivation steps were shown horizontally in Activity 5.2, but here they should be shown vertically.

(d) Now you are ready to solve the orienteer's problem, by carrying out the three steps of the Optimisation Procedure, as follows.

If you have difficulty in following these instructions, then you may like to look ahead to the first note in the Comment below, which shows what should eventually appear on the screen.

Step 1: Find the stationary points of f

Use the $\frac{d}{dx}$ icon on palette 1 and **Evaluate Symbolically** from the **Symbolic** menu to differentiate the expression to be optimised, which is

$$0.125 \cdot \sqrt{1 + x^2} + 0.0625 \cdot (2 - x).$$

You can use the key sequence

```
(0.125*\1+x^2[↑][↑][↑][↑]+0.0625*(2-x))
```

to enter this. The \ (backslash) keystroke creates the square root sign. Remember to enclose the expression to be differentiated in outer brackets.

The derivative should appear underneath the original expression. If not, then review the instructions above about 'Derivation Format...'.

To find the stationary points, select an occurrence of x in the expression you have just found for the derivative (put the blue bar cursor next to one of the xs) and choose **Solve for Variable** from the **Symbolic** menu. You should see two solutions, $-0.577\,350\ldots$ and $0.577\,350\ldots$, within a column vector. These are the values of x where the derivative is zero; that is, the stationary points.

To obtain the blue bar cursor, click on the expression (anywhere will do), then press the 'Down Arrow' key [↓] repeatedly until the cursor appears. You can use the left [←] and right [→] arrow keys to move the cursor into position next to an x.

Step 2: Evaluate f at each of the relevant points

Use Mathcad to evaluate the original expression at the two endpoints of the interval, $x = 0$ and $x = 2$, and at the stationary point between them.

There are several ways of doing this. Probably the most efficient is to define the function

$$f(x) := 0.125 \cdot \sqrt{1 + x^2} + 0.0625 \cdot (2 - x).$$

You can then evaluate the function for any given value of x.

Copy and paste is a handy way of avoiding the need to retype awkward or lengthy expressions, so it is worth remembering.

Rather than create the whole function definition from scratch, you can take advantage of Mathcad's copy and paste facilities. You need to create the left-hand side of the definition, $f(x) :=$ (that is, type `f(x):`), but then, for the right-hand side, you can copy the expression that you differentiated in Step 1. To do this, select it (click anywhere in the expression and press the 'Up Arrow' key [↑] until the whole expression is enclosed within the blue selection box) and choose **Copy** from the **Edit** menu. Then paste this expression into the function definition, by first clicking on the placeholder following $f(x) :=$ to select it, and then choosing **Paste** from the **Edit** menu.

Once you have defined the function $f(x)$, evaluate f at the interval endpoints, $x = 0$ and $x = 2$, and at the stationary point; for example, type `f(0)=` to evaluate $f(0)$. It is sufficient to input the value for the stationary point to three decimal places, $x = 0.577$, or you can use copy and paste to input all twenty digits given in the solution if you wish!

Step 3: Choose the optimum value

Remember that the orienteer's problem is a minimisation problem.

Identify the least of the three function values that you found. This is the minimum value of $f(x)$ within the given interval.

You could finish at this point, but it is a good idea to record your conclusions in the document. For example, you could enter the text

```
The least value of f(x) on [0,2] is ?  (at x=?).
```

replacing the question marks with the values that you found. You can add more text if you wish, but the Optimisation Procedure for the orienteer's problem is now completed.

(e) Finally, save your file, by choosing **Save As...** from the **File** menu. You will need to give it a suitable name that indicates the contents, for example, *my121C1-orienteer*. This will help you to organise and identify your files.

Solutions are given on page 31.

Comment

- The Mathcad document should now look something like this:

  Optimisation - the orienteer's problem

  Find the distance x that minimises the time taken to run f(x)

  Step 1 Find the stationary points of f

  $$\frac{d}{dx}\left[0.125\sqrt{1+x^2}+0.0625(2-x)\right]$$

  yields

  $$\frac{.125}{\sqrt{1+x^2}}\cdot x-6.25\cdot 10^{-2}$$

  has solution(s)

  $$\begin{pmatrix} -.57735026918962576451 \\ .57735026918962576451 \end{pmatrix}$$

  Step 2 Evaluate f at the endpoints x = 0 and x = 2, and at the stationary point x = 0.577

  $f(x) := 0.125\sqrt{1+x^2}+0.0625(2-x)$

  $f(0) = 0.25$ $\quad f(2) = 0.28$ $\quad f(0.577) = 0.233$

  Step 3 Choose the optimum value

  The least value of f(x) on [0, 2] is 0.233 (at x = 0.577)

Your document should contain the calculations shown but may have different text

- When an expression does not contain an equals sign, **Solve for Variable** finds the values of the selected variable that make the expression equal to zero. Note that Mathcad omits the zero before the decimal point in the solutions.

Mathcad notes

- Mathcad provides keyboard alternatives for most of the menu commands and features used above, as follows:

new blank document	type [F7] (function key 'F7', not the separate keys 'F' and '7');
$\frac{d}{dx}$ icon (on palette 1)	type ? (a question mark, given by [Shift]/);
create text region	type " (a double-quote, given by [Shift]2);
evaluate symbolically	type [Shift][F9];
copy	type [Ctrl]c;
paste	type [Ctrl]v.

- When working symbolically, a decimal point in the input expression triggers a decimal result, with up to twenty decimal places! However, when working numerically (evaluating an expression using =), the number of decimal places displayed depends on the 'Displayed Precision' (**Math** menu, **Numerical Format...**) specified for the result. (By default, the results of numerical calculations are displayed to three decimal places.)

Now close the Mathcad file that you have created and saved.

A traffic planning model

Traffic planners wish to set up a mathematical model to describe how the volume flow rate of traffic (that is, the number of vehicles which pass a fixed point in a given time) varies with the average velocity of vehicles along a single lane of a road. The eventual purpose of this model is to advise on how traffic flow can be maximised

The planners assume that each vehicle moves at a constant velocity $v\ \mathrm{m\,s^{-1}}$. On the basis of a subsidiary model and many observations, they estimate that, on average, each driver maintains a distance $v + 0.02v^2$ metres between the front of their vehicle and the back of the vehicle immediately ahead. The average length of a vehicle is estimated to be 5 metres. The model therefore represents the situation as shown in Figure 5.2.

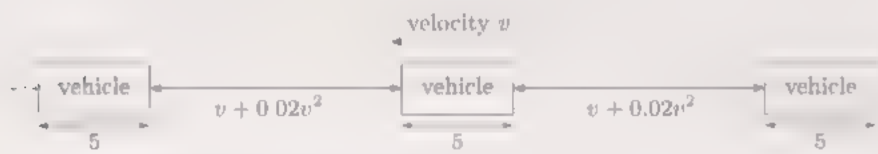


Figure 5.2 Representation of traffic flow

The distance between the fronts of successive vehicles is $5 + v + 0.02v^2$ metres. Each vehicle, travelling at velocity $v\ \mathrm{m\,s^{-1}}$, covers this distance in $(5 + v + 0.02v^2)/v$ seconds. It follows that the volume flow rate $f(v)$ of traffic (in vehicles per second past a fixed point) is given by

$$f(v) = \frac{v}{5 + v + 0.02v^2}.$$

A velocity of $35\ \mathrm{m\,s^{-1}}$ is about 126 kilometres per hour or 78 miles per hour.

This formula is to apply for $0 \leq v \leq 35$.

The traffic planners seek the maximum value of $f(v)$ for v in the interval $[0, 35]$. If such a maximum can be found, it can be used to provide an advised speed of travel on the road.

In the next activity you are asked to create a Mathcad document once more, to find both the greatest value of the traffic flow rate function $f(v)$ and the value of v for which this occurs.

Activity 5.4 Maximising the traffic flow rate

You do not need to load the symbolic processor or set the derivation format if you are continuing directly from Activity 5.3.

(a) Create a new blank document, load the symbolic processor, and set the derivation format as described in Activity 5.3(c).

(b) Enter a title at the top of your document, for example, `Maximising the traffic flow rate`.

(c) Solve the traffic flow rate problem by finding the maximum value of the function

$$f(v) = \frac{v}{5 + v + 0.02v^2}$$

Remember that this is a maximisation problem

on the interval $[0, 35]$ and the corresponding value of v. Do this by carrying out the three steps of the Optimisation Procedure, just as you did in Activity 5.3(d).

(d) State your answer in a form which is appropriate in the context of the traffic planning model.

Solutions are given on page 31.

Comment

If you have defined a function $f(v)$ for the volume flow rate of traffic in your document then you may wish to display a graph of it. You can do so by defining a suitable graph range, for example, $v := 0, 0.1 .. 35$, and then plotting $f(v)$ against v. While the graph confirms the maximum value at $v \simeq 16$, it also shows that there is little difference in the value of $f(v)$ between $v = 10$ and $v = 20$.

Save your file, for example as my121C1-traffic. Then close this file.

Chapter C2, Section 5
Integration with the computer

There is only one prepared Mathcad file for this section, and that comes towards the end and is optional. As in the later computer activities for Chapter C1, you will, for the most part, be creating your own documents here and using Mathcad directly. Subsection 5.1 shows how to find indefinite integrals in Mathcad, while Subsection 5.2 covers definite integrals.

5.1 Finding indefinite integrals

In Mathcad, indefinite integrals are found using the $\int$ operator. Unlike the $\frac{d}{dx}$ operator, the $\int$ operator can be used only symbolically, to find an algebraic expression for an integral of a given function. It cannot be used numerically.

In this subsection you are invited to find indefinite integrals for a variety of functions. The first activity provides an introduction to finding indefinite integrals using Mathcad.

Activity 5.1 Indefinite integrals

In this activity you will use Mathcad to find the indefinite integral of x^2.

If necessary, see Chapter C1, Activity 5.3(a) and (c) on page 9 of this computer book. However, note that the derivation steps here are to be shown horizontally.

Be careful not to confuse the palette 3 icon $\int$ with the palette 1 icon $\int_a^b$, which is used for finding definite integrals (as you will see later).

(a) Create a new blank Mathcad document and load the symbolic processor. Then set the derivation format to show derivation comments and to display the derivation steps horizontally.

(b) Enter the $\int$ operator in your document, either by clicking on the palette 3 icon $\int$, or by using the keyboard shortcut `[Ctrl]i`. (To reveal palette 3, click on the number button '1' at the top of palette 1 on the left-hand side of the Mathcad window, then on the number button '2'.)

(c) Enter the expression to be integrated, x^2, in the first placeholder after the integral sign. (This expression is called the integrand.) Then enter the variable of integration, x, in the placeholder after the 'd'.

(d) Use the 'Up Arrow' key `[↑]` to select the entire expression (including the integral sign). Then choose **Evaluate Symbolically** from the **Symbolic** menu, or use the keyboard shortcut `[Shift][F9]`. Check that the answer provided by Mathcad is what you expect.

(e) Now go through the same procedure to evaluate the integral $\int u^2\,du$.

It is a good idea to insert a title at the top of any document that you save. If you need to create space for this, then do so by putting the red cross cursor at the top of the document and using **Ins/Del Blank Lines...** from the **Edit** menu.

(f) If you wish to save your work, then select the **File** menu and use **Save As...** to name and save your document.

Comment

- Notice that the $\int$ operator in Mathcad gives only *an* integral of the integrand. It does not give *the indefinite* integral because it does not add an arbitrary constant.
- The outcomes from integrating $\int x^2\,dx$ and $\int u^2\,du$ demonstrate that the form of the indefinite integral depends on the nature of the function being integrated but not on the choice of symbol for the variable of integration.

Mathcad notes

- Remember that symbolic evaluation occurs only when you select something and choose the command from the menu, or use the keystroke `[Shift]` `[F9]`. The result of a symbolic evaluation is *not* updated automatically if you just change the original expression. In part (e), for example, it is not sufficient simply to edit the integral expression $\int x^2\,dx$, replacing each 'x' by a 'u'
- If you enter the expression $\int x^2\,dx$ and then press the = key, the error message 'Symbolic operator only' appears, whether or not you have previously specified a value for x. This is as it should be, since the = key is used for numerical evaluation, which does not make sense in the context of finding an indefinite integral.
- You cannot find an integral of a function f by first defining f and then evaluating the expression $\int f(x)\,dx$. If you try it, then Mathcad will simply repeat the expression. There has to be an actual algebraic expression in the integrand placeholder for symbolic integration to be carried out.

In each of the remaining activities in this subsection, you can either continue using the document from the previous activity, or close that file and create a new blank document. You will need to decide whether you wish to save your work.

For each activity, you will also need to ensure that Mathcad is prepared to carry out symbolic calculations. The symbolic processor should be loaded, and the derivation format set to show derivation comments and to display the derivation steps horizontally.

In the next activity, you are asked to use Mathcad to find the indefinite integrals of two functions that you integrated by hand in the main text, and of a third function which you could also integrate by hand.

Activity 5.2 Finding indefinite integrals

Use Mathcad to evaluate each of the following indefinite integrals. (If necessary, refer to the instructions for evaluating an integral in Activity 5.1(b) (d).) Enter each integral on a new line.

If you arrive at incorrect answers, then see the Mathcad note below

(a) $\displaystyle\int\left(\frac{1}{x}+e^{3x}\right)dx \quad (x>0)$

(b) $\displaystyle\int\left(\frac{3}{y^4}+5\sin(5y)\right)dy \quad (y>0)$

(c) $\displaystyle\int(a+\cos(ax))\,dx \quad$ (where a is a non-zero constant)

Solutions are given on page 32.

Comment

- After the '$+c$' has been added, the Mathcad answers to parts (a) and (b) agree with those obtained in the main text, and that for part (c) is identical to the answer found by applying Table 1.1. In part (a), recall that $\exp(3x)$ is an alternative way of writing e^{3x}.
- Note that, while Mathcad gives answers here that agree with those obtained by hand, there is no way of entering the additional constraints on the variables that accompany the integrals, for example, the condition $x > 0$ in part (a). For this reason, it is necessary to bear in mind that the symbolic manipulations performed by Mathcad might not be valid in all circumstances, and that the output should be interpreted with care.

Mathcad notes

You need to type **3*x** for $3x$ in part (a), **5*sin(5*y)** for $5\sin(5y)$ in part (b) and **a*x** for ax in part (c), because all products must be made explicit in Mathcad. For example, if you type **3x** rather than **3*x** then Mathcad will assume that you have entered a single variable name, '$3x$', rather than the product of the number 3 and the variable x.

The next activity contains four integrals which you evaluated by hand in Section 2, followed by three further integrals (in parts (e)–(g)) that cannot be evaluated by hand simply on the basis of what is given in the chapter.

Activity 5.3 Further integrals

Use Mathcad to evaluate each of the following indefinite integrals.

(a) $\int (x-3)(x-1)\,dx$ (b) $\int \frac{2x-3}{\sqrt{x}}\,dx$

Note that $\sin^2 x$ should be input as `sin(x)^2`.

(c) $\int \sin^2 x\,dx$ (d) $\int \frac{x}{x^2+1}\,dx$

(e) $\int ue^{3u}\,du$ (f) $\int x^2\ln(5x)\,dx$ (g) $\int \frac{1}{\sqrt{9-t^2}}\,dt$

Solutions are given on page 32.

Comment

- The answers obtained from Mathcad in parts (a)–(d) are equivalent to the answers found by hand. However, they are not always given in an identical form, and in part (c) some algebra is required to show the equivalence of the two forms.
- It is possible to use the $\frac{d}{dx}$ operator to differentiate symbolically each of the integrals obtained in this activity. This leads to an expression equivalent to the original integrand in each case, as expected. However, the output obtained is not always identical in form to that of the original integrand, and again some algebra is sometimes required to show the equivalence of the two forms.
- Recall that **Simplify** from the **Symbolic** menu can sometimes be used to simplify to some extent an expression obtained using **Evaluate Symbolically**. In many cases **Factor Expression**, also from the **Symbolic** menu, can be applied in a similar way.

Mathcad notes

The Mathcad command **Integrate on Variable**, available from the **Symbolic** menu, provides an alternative approach to finding a formula for an integral of a function. This is described in *A Guide to Mathcad*. However, it is usually better to use the $\int$ operator and **Evaluate Symbolically**, because this results in clearer Mathcad documents.

Parts (e)–(g) of Activity 5.3 show that Mathcad can extend your 'integration reach' beyond the types of functions that you have so far learnt how to integrate by hand. However, Mathcad cannot integrate every function. For example, if you ask Mathcad to evaluate symbolically either of the indefinite integrals

$$\int e^{-t^3}\,dt \qquad \text{and} \qquad \int \sqrt{\sin x}\,dx,$$

the response is for Mathcad to repeat the given integral without alteration (except for the replacement of e^{-t^3} by $\exp(-t^3)$ in the first case). This is how Mathcad responds when it cannot find an integral. This need not be due to a shortcoming of Mathcad. In the two cases here, no expression is known for either integral in terms of standard functions.

5.2 *Definite integrals, areas and summations*

You will need Mathcad file 121C2-01 for (optional) Activity 5.8, later in this subsection. First, however, you are invited to create your own Mathcad documents as in the previous subsection, but now to find definite rather than indefinite integrals.

In Mathcad definite integrals are found using the $\int_a^b$ operator. This operator can be used either symbolically or numerically. If the operator is used *symbolically*, then Mathcad finds an algebraic expression for an integral, evaluates this expression at the upper and lower limits of integration, and subtracts the second value from the first to find the answer. This is the same as the usual approach to finding definite integrals by hand. If the operator is used *numerically*, then Mathcad does not find an algebraic expression, but instead uses a numerical method to find an approximate value for the definite integral.

In each of Activities 5.4–5.7 below, as in Subsection 5.1, you can either continue using the document from the activity before, or close that file and create a new blank document. You will need to decide whether you wish to save your work.

For each activity, you will also need to ensure that Mathcad is prepared to carry out symbolic calculations. The symbolic processor should be loaded, and the derivation format set to show derivation comments and to display the derivation steps horizontally.

Activity 5.4 introduces you to the symbolic use of the $\int_a^b$ operator.

Activity 5.4 Evaluating definite integrals symbolically

In this activity you will use Mathcad to evaluate the definite integral

$$\int_2^3 \frac{1}{x}\,dx.$$

(a) Enter the $\int_a^b$ operator in your document, either by clicking on the icon in palette 1, or by using the keyboard shortcut & (the ampersand sign, given on the keyboard by [Shift]7).

(b) Enter the integrand, the variable of integration, and the upper and lower limits of integration in the appropriate placeholders.

The [Tab] key provides a good way of moving around the placeholders.

(c) Select the entire expression, including the integral sign, and choose **Evaluate Symbolically** from the **Symbolic** menu, or use the keyboard shortcut [Shift][F9]. You should obtain the answer

$$\ln(3) - \ln(2).$$

(d) Leave the answer selected, and press the = key. You should obtain the answer 0.405, which is the value of $\ln 3 - \ln 2$ to three decimal places.

Comment

Evaluating symbolically an expression involving the $\int_a^b$ operator gives an expression which is an *exact* answer (unless the original expression contains a decimal point; see the second Mathcad note below). In the example in this activity the expression is $\ln(3) - \ln(2)$. You can display the decimal value of such an expression by evaluating it numerically. This is done by selecting the expression, and then pressing the = key. (If the expression has just been obtained as a result of a symbolic calculation, then it will already be selected.)

Mathcad notes

◇ When you evaluate numerically an expression in Mathcad, the number of decimal places displayed is determined by the value of 'Displayed Precision'. The default value of this is 3, but you can change it by choosing **Numerical Format...** from the **Math** menu.

◇ If a Mathcad expression involving the $\int_a^b$ operator has a decimal point in any constant in the integrand, or in *both* limits of integration, then evaluating the expression symbolically gives a decimal answer with up to twenty decimal places. (Such an answer is unaffected by the value of 'Displayed Precision'.) For example, evaluating symbolically the Mathcad expression

$$\int_2^3 \frac{1.0}{x}\,dx \quad \text{yields} \quad .40546510810816438198.$$

The next activity provides further practice in using Mathcad to evaluate definite integrals symbolically.

Activity 5.5 Further definite integrals (symbolically)

Each of parts (a)–(d) below gives a definite integral that you were asked to evaluate by hand in the main text. In each case, use Mathcad to evaluate the definite integral symbolically to obtain an exact answer, and then evaluate this answer numerically, to display it as a decimal value.

(a) $\int_0^2 e^t\,dt$ (b) $\int_0^{\pi/4} (\cos(5x) + 2\sin(5x))\,dx$

(c) $\int_1^2 \frac{6}{u^2}\,du$ (d) $\int_0^{\pi} e^t \sin t\,dt$

Remember that π can be obtained from palette 1, or by typing [Ctrl]p

Solutions are given on page 32.

In the next activity you will see an example of a definite integral that Mathcad is unable to evaluate symbolically. When this happens it is often worth attempting to evaluate the integral numerically, and the activity shows you how to do this.

Activity 5.6 Evaluating definite integrals numerically

(a) Use Mathcad to try to evaluate symbolically the definite integral

$$\int_0^1 e^{-t^3}\,dt.$$

You should obtain the warning message 'No closed form found for integral'. This means that Mathcad has been unable to calculate an algebraic expression for an integral of the integrand, and so it cannot evaluate symbolically the given definite integral.

At the end of Subsection 5.1 it was pointed out that Mathcad cannot evaluate the indefinite integral

$$\int e^{-t^3}\,dt.$$

Click on 'OK' to remove the message.

(b) Evaluate the definite integral numerically, as follows. It should still be selected; if it is not, then select it by putting the blue bar cursor anywhere within it, or by surrounding all or part of the expression with the blue selection box. Then type =.

You should find that the answer 0.808 is displayed.

Comment

◇ In general, to evaluate a definite integral numerically, you should enter it in the same way as for symbolic evaluation, then select it and type =.

◇ The reason why some definite integrals can be evaluated numerically but not symbolically in Mathcad is that symbolic evaluation requires Mathcad to find an algebraic expression for an integral, whereas numerical evaluation involves the use of a numerical algorithm. The answer obtained from this algorithm is an approximation, though usually an accurate one.

Activity 5.8 indicates a possible basis for such an algorithm.

Mathcad notes

On rare occasions, the numerical method used by Mathcad for evaluating definite integrals fails to produce a value. In such a case, the integral is marked with the error message 'not converging'.

The next activity gives you further practice in using the $\int_a^b$ operator numerically.

Activity 5.7 Further definite integrals (numerically)

The first two steps here are identical to those for symbolic integration. These steps would also have been required for the definite integral in Activity 5.6(b), had the expression not previously been entered.

(a) Evaluate numerically the definite integral $\int_0^1 t^2\,dt$, as follows.

(i) Either click on the palette 1 icon $\int_a^b$ or type & to insert the definite integral operator and its placeholders.

(ii) Fill in the four placeholders appropriately.

(iii) The cursor should be on the expression; if not, select some part of the integral. Then evaluate the expression numerically, by typing =.

(b) Evaluate numerically the definite integral $\int_{-1}^{1} \sqrt{1-x^2}\,dx$.

Solutions are given on page 32.

Comment

These two definite integrals can be evaluated either numerically or symbolically. As you can check, symbolic evaluation gives $\frac{1}{3}$ for (a) and $\frac{1}{2}\pi$ for (b).

You have seen that in Mathcad many definite integrals can be evaluated either symbolically or numerically. This raises the question of which it is more appropriate to invoke in any given situation. If you want an exact answer (so you can see where constants such as π feature in it, for example), or if a general result is required, such as a formula for

$$\int_0^1 \cos(ax)\,dx \quad \text{(where } a \text{ is a non-zero constant)},$$

then use the symbolic approach. If you simply want a number that is an accurate value for the definite integral, then numerical evaluation should suffice. If you want both an exact value and a decimal value for the answer, then you can first evaluate the definite integral symbolically and afterwards press =.

For example, the values of all the definite integrals in Activity 5.5 can be found satisfactorily using numerical integration.

Integration as a limit of summations

The remainder of this subsection will not be assessed.

In Subsection 4.2, you saw that a definite integral could be approximated as closely as required by a finite sum. This was demonstrated in particular for the case in which the value of the definite integral gives the area beneath the graph of a function, and each finite sum represents the overall area of a set of rectangles. Each rectangle is based upon a subinterval, and the approximation to the definite integral improves as the number of subintervals is increased.

We assume (as in the main text) that $f(x)$ takes only non-negative values in the interval $[a, b]$.

For the area beneath the graph of the function $f(x)$ between $x = a$ and $x = b$, which is given exactly by the definite integral

$$\int_a^b f(x)\,dx,$$

the approximation based on N subintervals is

$$\sum_{i=0}^{N-1} hf(a+ih) = h\sum_{i=0}^{N-1} f(a+ih), \quad \text{where } h = \frac{b-a}{N}$$

See equation (4.1) in Chapter C2, Subsection 4.2.

The remaining (optional) activity in this section asks you to use the prepared Mathcad file 121C2-01 to explore the relationship between these approximations to the area under the curve and the definite integral itself.

Activity 5.8 Integration as a limit of summations (Optional)

(a) Open Mathcad file 121C2-01 and read through the document, which consists of a single page. The definite integral being approximated here is

$$\int_0^{40} 15\sqrt{\sin\left(\frac{\pi x}{40}\right)}\,dx,$$

whose value was sought by a process of successive approximation in Subsection 4.2.

The value of this integral is the area on the graph beneath the red curve. The value calculated for A is an estimate for this area, using approximation by rectangles based on N subintervals. The value of N is initially set to 4.

Investigate the effect of increasing the number of subintervals, N. Use in turn the following values for N:

20, 50, 100, 500, 1000, 5000, 10 000.

Compare the corresponding values obtained for A with those in the right-hand column of the table below.

Number of subintervals	Sum of areas of rectangles
4	402.27
20	452.71
50	456.41
100	457.21
500	457.62
1 000	457.64
5 000	457.66
10 000	457.66

This is Table 4.1 in Chapter C2, Subsection 4.2.

(b) You may like to use the file for other definite integrals, to investigate the behaviour of area estimates A as N is increased. For example, you could investigate these estimates for the definite integral

$$\int_0^1 e^{-x^3}\,dx,$$

by first editing the file so that $f(x) = e^{-x^3}$ and $b = 1$.

If you have time, look also at the behaviour of the corresponding area estimates for the definite integral

$$\int_0^{\pi/4} \tan x\,dx$$

Comment

- You should find that the numerical values obtained for A in part (a) match to two decimal places the values given in the right-hand column of the table. These area estimates appear to tend to the limit 457.66 (to two decimal places), which is also the numerical value provided by Mathcad for the definite integral.
- The values of the definite integrals suggested in part (b) are

$$\int_0^1 e^{-x^3}\,dx = 0.808 \quad \text{and} \quad \int_0^{\pi/4} \tan x\,dx = 0.347,$$

each to three decimal places. In each case, the estimates A converge to this value as N increases.

In the first example, the area estimates are greater than the value given by the definite integral for the area under the curve, while in the second example they are smaller. You can see this illustrated on the graph by setting small values of N ($N \leq 20$, say). In the first example, the tops of the rectangles all lie above the curve, while in the second example they all lie below.

Mathcad notes

- The summation sign is obtained from the palette 1 icon $\sum_{n=1}^{m}$ or by typing `[Ctrl]$` (you have to press the three keys `[Ctrl]`, `[Shift]` and 4 together).
- Note that the definite integral is set up in this file with integrand $f(x)$, where the function $f(x)$ is defined earlier in the document. This presents no problem for *numerical* evaluation of the definite integral. However, it is not possible to perform *symbolic* evaluation of a definite integral in this form. This is in line with previous Mathcad notes concerning symbolic evaluation and functions defined earlier in a document.

See the third Mathcad note for Activity 5.1 and the first Mathcad note for Chapter C1, Activity 5.2.

- The rectangles are filled in by drawing zig-zag lines very close together. On the screen these lines give the appearance of a solid block of colour, but the tiny gaps between the lines may become apparent if printed.

Now close Mathcad file 121C2-01.

Chapter C3, Section 4
Differential equations with the computer

In this section you will use the computer to see how the information contained in a differential equation can be displayed graphically, and how differential equations can be 'solved' numerically and graphically, even where no formula for the solution can be found.

There are two prepared Mathcad files associated with this section. The first draws direction fields and plots solution curves (graphs of solutions) for first-order differential equations, and the second illustrates how a numerical method for obtaining the solution to an initial-value problem works in practice.

This numerical method can be applied to any differential equation of the form

$$\frac{dy}{dx} = f(x, y),$$

where f is a known function of two variables. This form includes the types of differential equation considered in the main text, but also others.

As special cases, f may be a function of x alone or of y alone.

Mathcad does not contain any symbolic facilities for solving differential equations directly, to find a formula for the solution. However, as you saw in Chapter C2, Mathcad can be used to find integrals, and finding integrals is the main constituent of the two methods for solving first-order differential equations that are introduced in the main text (direct integration and separation of variables). Hence Mathcad can be used indirectly to help solve some first-order differential equations. Mathcad can also be applied to differentiate a solution that you have already found to check that it is indeed a solution to the given differential equation.

Differentiation was the subject of Chapter C1.

In this section, however, Mathcad will be used numerically and graphically rather than symbolically.

4.1 Direction fields and solution curves

The differential equation

$$\frac{dy}{dx} = x + y$$

cannot be solved, using the methods of this chapter, to give an equation relating x and y. However, the direction field of this differential equation provides enough information to indicate the different types of solution curve that occur. In Activity 4.1 you will see this direction field drawn by Mathcad, and will be able to ask Mathcad to plot the solution curve through any point of your choice.

Activity 4.1 The differential equation $dy/dx = x + y$

Open Mathcad file 121C3-01. The document opens with the direction field of the differential equation

$$\frac{dy}{dx} = x + y$$

drawn for a grid of points with integer coordinates in the (x, y)-plane, for values of x between -5 and 5, and values of y between -2 and 4. For each such point (x, y) a line segment is plotted through the point, and the slope of this line segment is $f(x, y) = x + y$.

The initial condition is $y(x_0) = y_0$.

(a) Set S to 1, so that a solution curve is plotted through the point $(0, 0)$. Briefly describe the curve obtained, or make a small sketch of it. Now change the value of y_0, to obtain a solution curve through each of the following points in turn:

$$(0, 1), \quad (0, 2), \quad (0, -1), \quad (0, -2).$$

In each case, note down a brief description of the curve or make a small sketch of it.

(b) Can you group the solution curves seen in part (a) into distinct types of behaviour? How many different types of behaviour are there?

For each of the following points, try to predict from looking at the direction field which type of behaviour the solution curve through the point will exhibit:

$$(-3, -1), \quad (-1, 0), \quad (4, 2).$$

Then, by making a suitable choice of values for x_0 and y_0, use Mathcad to plot the corresponding solution curve and to confirm your prediction.

Solutions are given on page 33.

Comment

The line segments of the direction field give a good indication of where the solution curves lie.

Mathcad notes

- A definition for a function of two variables is created in the same way as that for one variable. For example, to define the function $f(x, y) = x + y$ in Mathcad, you could type `f(x,y):x+y`.
- The calculations used to draw the direction field and solution curve are 'hidden' off the page of the Mathcad document to the right. The area beyond the right-hand margin and edge of a Mathcad page (which are marked by solid and dotted vertical lines, respectively) can be used in the same way as the rest of the document. You can place mathematical expressions, text or graphs there. You do not need to look at the calculations in this document, but in general, you can view what is in the 'hidden' area by using the horizontal scroll bar to move to the right.

In the next activity you are asked to use Mathcad to plot the direction fields for two other first-order differential equations. In each case, you can try to visualise from the direction field how the solution curves behave, before plotting some of these curves.

Activity 4.2 Direction fields and solution curves

You should still be working with Mathcad file 121C3-01

(a) Investigate the direction field and solution curves for the first-order differential equation

$$\frac{dy}{dx} = e^{\cos x} - 1,$$

as follows.

(i) First set the variable S to zero, so that no solution curve is plotted. Then enter the right-hand side for this differential equation into the definition for $f(x, y)$.

(ii) Try to predict from the direction field where the solution curves lie. What types of behaviour will the solution curves exhibit?

(iii) Set the variable S to 1 to display a solution curve, and then try different values for x_0 and y_0 to confirm the predicted behaviour of solution curves. Use your observations to describe the behaviour of the solution curves for the differential equation.

(b) Now follow the same procedure as in part (a) to investigate the direction field and solution curves for the first-order differential equation

$$\frac{dy}{dx} = xy + 1.$$

However, in this case, start by altering the scope of the grid in the y-direction, by setting $Y1 := -5$, $Y2 := 5$ and $q := 10$.

Solutions are given on page 33.

Comment

◇ The differential equation in part (a) is of the form $dy/dx = f(x)$, and so can be solved in principle by direct integration. However, it turns out not to be possible to find an algebraic expression for

$$\int (e^{\cos x} - 1)\, dx.$$

The presence of the $\cos x$ means that the slopes of the direction field repeat at horizontal intervals of 2π. (The slopes are also invariant in the vertical direction, for any given choice of x, as noted in the solution.) The alternate positive and negative slopes indicate that solution curves will undulate. The overall increasing trend is not so obvious from the direction field, but might be expected because the magnitude (steepness) of the slopes appears to be greater where the slopes are positive than where they are negative.

◇ The differential equation in part (b) cannot be solved by the methods of this course. Using Mathcad to plot the direction field provides a good approach to visualising the types of solution curve for such a differential equation.

Now close Mathcad file 121C3-01.

4.2 Euler's method

In Subsection 4.1 you saw how a direction field can be used to visualise the information provided by a first-order differential equation. You also saw solution curves plotted on top of the direction field. It is straightforward to plot such curves for a first-order differential equation whose solutions can be expressed in terms of a simple algebraic formula. However, even where no such formula can be found, approximate solution curves for a differential equation can still be plotted. These graphs are based on a sequence of numerical estimates for solution values, which constitute a numerical solution to the differential equation. In this subsection you will see how such a numerical solution can be obtained.

This was the case for each of the differential equations in Activity 4.2.

The procedure to be described below, for obtaining an approximate numerical solution to a first-order differential equation, is known as *Euler's method*. To see how the solution is built up step by step, it is illuminating to consider the corresponding graphical construction. This involves forming a connected chain of line segments, each of which has a gradient given by the slope of the direction field at the left-hand end of the line segment, as shown in Figure 4.1.

Recall that a direction field provides a slope value at every point within a given region, and not just at the particular grid points displayed on a graph.

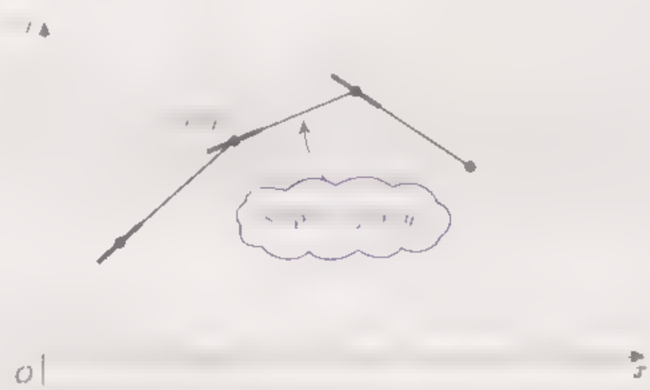


Figure 4.1 Graphical construction for Euler's method, where $dy/dx = f(x, y)$

You will see this idea explained in greater detail in the next activity.

Activity 4.3 Introducing Euler's method

Open Mathcad file 121C3-02, read the introduction and move to page 2. Here a direction field has been set up for the function $f(x, y) = e^{\cos x} - 1$; that is, for the differential equation

$$\frac{dy}{dx} = e^{\cos x} - 1.$$

You studied this differential equation in Activity 4.2(a).

Also, an initial condition is specified, as

$$y = 0 \text{ when } x = 0; \quad \text{that is,} \quad y(0) = 0.$$

In the file, the initial values of x and y are denoted respectively by x_0 and y_0.

Note, to the right of the graph, that the value $f(x_n, y_n) = 1.718$ is displayed, where $n = 0$. This is (to 3 d.p.) the value of

$$f(0, 0) = e^{\cos 0} - 1 = e - 1,$$

which is the slope of the direction field at the starting point $(x_0, y_0) = (0, 0)$. This starting point is denoted on the direction field by a small blue box, and the direction of the direction field at this point is coloured magenta.

You will now see how an approximate solution to this initial-value problem can be built up graphically, step by step. Ensure that all of page 2 from the heading 'Solution curve' to the bottom of the graph is visible on your screen.

(a) Change the value under 'Number of steps' in turn to $N = 1$, $N = 2$, $N = 3$, $N = 4$ and $N = 5$. In each case, observe the effects that the change in value causes.

(b) Observe the effect of changing, in turn, the step size to $h = 0.5$ and the number of steps to $N = 10$.

(c) Observe the effect of changing the initial values. For example, set $x_0 = 1$ and $y_0 = 2$.

Comment

Comment on part (a) for $N = 1$

◇ When 'Number of steps' is changed to $N = 1$, the first segment of the approximate solution curve is drawn, from the starting point $(x_0, y_0) = (0, 0)$ to the point $(x_1, y_1) = (1, 1.718)$. The coordinates of these points appear in the tables to the right of the graph. The slopes of the direction field at these points, $f(x_0, y_0) = 1.718$ and $f(x_1, y_1) = 0.717$, are also shown.

All of the values are shown to three decimal places. The slope of the direction field at (x_1, y_1) is now shown by a line segment on the graph, coloured magenta.

The two broken blue lines that appear on the graph are temporary construction lines. They illustrate how the first segment of the solution curve is drawn as the hypotenuse of a right-angled triangle. The base (run) of this triangle is equal to the *step size*, which is specified above the graph as $h = 1$. The triangle is constructed, as shown in Figure 4.2, so that *the gradient of its hypotenuse is equal to the gradient given by the direction field at its left-hand vertex.* This gradient is $f(x_0, y_0)$. On the other hand, the gradient (slope) of the hypotenuse is given as usual by the rule 'slope equals rise over run', where the run is the step size, h. The rise, which is the height of the triangle, is therefore given by

$$\text{rise} = \text{run} \times \text{slope} = hf(x_0, y_0).$$

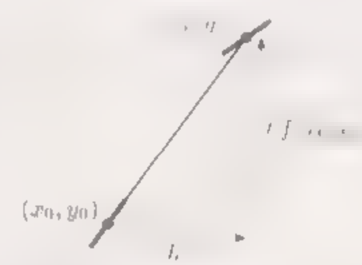


Figure 4.2 First segment

The coordinates (x_1, y_1) of the top vertex of the triangle can then be calculated from the coordinates (x_0, y_0), the rise and the run. This gives

$$\begin{aligned}(x_1, y_1) &= (x_0 + \text{run},\ y_0 + \text{rise})\\ &= (x_0 + h,\ y_0 + hf(x_0, y_0))\\ &= (0 + 1,\ 0 + 1.718) = (1,\ 1.718).\end{aligned}$$

Comment on part (a) for $N = 2$

◇ When the number of steps is changed to $N = 2$, the second segment of the approximate solution curve is added, from the point $(x_1, y_1) = (1, 1.718)$ to the point $(x_2, y_2) = (2, 2.435)$. The broken blue lines now illustrate the 'construction triangle' for this second line segment, as shown also in Figure 4.3. The base (run) of this triangle is again equal to the step size, $h = 1$, but the gradient of its hypotenuse is equal to the gradient given by the direction field at (x_1, y_1). This gradient is $f(x_1, y_1) = 0.717$, as indicated by the table on the right of the screen. The coordinates (x_2, y_2) are therefore given by

$$\begin{aligned}(x_2, y_2) &= (x_1 + \text{run},\ y_1 + \text{rise})\\ &= (x_1 + h,\ y_1 + hf(x_1, y_1))\\ &= (1 + 1,\ 1.718 + 0.717) = (2,\ 2.435).\end{aligned}$$

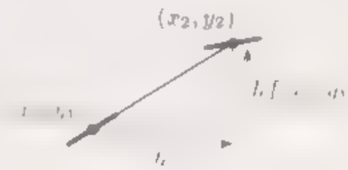


Figure 4.3 Second segment

Comment on part (a) for $N = 3$, $N = 4$ and $N = 5$

Note that the values of $f(x_n, y_n)$ given in the table are negative for $n = 2, 3$ and 4. These correspond to the negative slopes of the direction field at the corresponding points (x_n, y_n), as is clear from the graph.

◇ With two steps now completed, it should be fairly clear how the process continues when the number of steps is changed to $N = 3$, $N = 4$ and $N = 5$. The chain of line segments (which form the approximate solution curve) and the rows of corresponding values in the tables to the right of the graph build up one by one as N is incremented. Each time, the latest line segment is joined to the previous one, and its slope matches that of the direction field at its left-hand end

Mathematically, this means that in constructing the line segment from (x_n, y_n) to (x_{n+1}, y_{n+1}), we have

$$\begin{aligned}(x_{n+1}, y_{n+1}) &= (x_n + \text{run}, y_n + \text{rise}) \\ &= (x_n + h, y_n + hf(x_n, y_n)) \quad (n = 0, 1, 2, \ldots).\end{aligned}$$

Figure 4.4 $(n+1)$th segment

This is illustrated in Figure 4.4. In other words, the sequences x_n and y_n are determined by the pair of recurrence relations

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hf(x_n, y_n) \quad (n = 0, 1, 2, \ldots),$$

as shown to the right of the graph on screen, together with the specified values for x_0 and y_0. These recurrence relations encapsulate **Euler's method**.

Each value y_n is an estimate of the 'true solution' y at $x = x_n$; that is, y_n is an estimate of $y(x_n)$. Clearly, the sequence of estimates obtained depends on the choice of both the step size, h, and the number of steps, N.

Comment on part (b)

◇ With N steps of size h, Euler's method provides $N + 1$ solution estimates, spaced at regular horizontal intervals between the chosen starting value, x_0, and $x_0 + Nh$.

Note that, when starting at $(x_0, y_0) = (0, 0)$, either 5 steps of size 1 or 10 steps of size 0.5 provide a final solution estimate at $x = 5$. By halving the step size and doubling the number N of steps, their product Nh, and hence also $x_0 + Nh$, remains the same. Notice, however, that the solution estimates obtained at $x = 5$ are different: the final coordinates are $(x_5, y_5) = (5, 0.986)$ in the first case and $(x_{10}, y_{10}) = (5, 0.558)$ in the second. Halving the step size and doubling the number of steps improves the accuracy of the solution estimate. In the next activity you can investigate further how accuracy improves with reductions in step size.

Comment on part (c)

◇ Setting $x_0 = 1$ and $y_0 = 2$ produces a different chain of line segments, which is again an approximate solution curve for the differential equation, now with the initial condition $y(1) = 2$. (The same is true for other choices of initial values.) If h and N are unaltered from part (b), then the last row of values at the right of the screen corresponds to a point which lies outside the displayed graph region.

Activity 4.3 showed how Euler's method works. You saw a connected chain of line segments built up, one by one, as an approximation to a solution curve. However, this was a fairly inaccurate approximation. The construction utilises slope values provided by the direction field only at points which are a horizontal distance h apart, where the values assigned to the step size h in Activity 4.3 were first 1 and then 0.5. As a result, the approximate solution curve was based on very limited information.

More information can be extracted from the direction field by reducing the step size, provided that also the number N of steps is increased to maintain coverage of the x-values over which a solution is sought. The next activity shows that such use of extra information leads to improvements in accuracy, and that an estimate for the solution $y(x)$ at a particular chosen value of x can, in principle, be found to whatever accuracy is required.

Activity 4.4 Using Euler's method

Move to page 3 of the document and read the first paragraph. Then scroll down until all of the page from the heading 'Graphics of the numerical solution' to the bottom of the graph is visible on your screen.

You should be working with Mathcad file [illegible].

The page is set up to solve the same initial-value problem that was considered in Activity 4.3(a) and (b), namely,

$$\frac{dy}{dx} = e^{\cos x} - 1, \quad y(0) = 0.$$

The computation, using Euler's method, takes place from the starting value $x = x_0$ to the finishing value $x = xval$, where $xval$ can be specified by the user. The step size h can be chosen as before. However, in contrast to the situation in Activity 4.3, it is not possible here to vary independently the number of steps, N. Given a step size h, the number of steps to be used is calculated automatically from the condition that the final step is to reach $x = xval$. As a result, the value of y_N is always an estimate for the value $y(xval)$ of the true solution at $x = xval$.

In other words, if h divides exactly into $xval - x_0$, then $N = (xval - x_0)/h$.

(a) Change the value of $xval$ to 9 (from its starting value of 5). What changes on the screen, and what stays the same?

(b) We now seek an estimate for the value of $y(9)$ (the true solution value at $x = 9$). This is obtained by progressively reducing the step size.

First note the value of y_N that appears on screen (this is the estimate for $y(9)$ obtained with step size $h = 1$, and hence with 9 steps).

Now change the step size in turn to 0.5, 0.2, 0.1, 0.01, 0.001 and 0.0001. In each case, note the corresponding value for y_N. Use these values to estimate the value of $y(9)$ to one decimal place.

Solutions are given on page 34.

Comment

- As the step size h is decreased, the number of steps increases. With $h = 0.0001$, there are 90 000 steps, for which the calculation may take an appreciable time on your computer.
- With $h = 0.5$, the approximate solution curve still looks like a connected chain of line segments. However, with $h = 0.2$ the graph appears significantly more like a smooth curve, and this remains the case for smaller values of h. (The graph is still in fact made up of short line segments, but so is every graph drawn by Mathcad with the trace type set to 'lines'!)
- Just as the numerical estimates for $y(xval)$ appear to converge as the step size is decreased, so too do the approximate solution curves seem to converge towards a 'limiting curve' on the direction field. The graphs obtained provide increasingly close representations of the actual solution curve.

Euler's method, as illustrated in Activities 4.3 and 4.4, was used to plot the solution curves in file 121C3-01, used for Activities 4.1 and 4.2. There, the number of steps was set to $N = 1000$, and calculations were made to draw the line segments to the left, as well as to the right, of the initial point. In that way, the line segments gave the appearance of a smooth solution curve over the whole horizontal graph range.

If you are interested and have the time, then you might like to try the following optional activity, which involves a function $f(x, y)$ that depends on the dependent variable y as well as on the independent variable x. For this particular example, it is also possible to check the outcome against a formula for the solution.

Activity 4.5 Another initial-value problem (Optional)

You should still be working with page 3 of Mathcad file 121C3-02. Before doing anything else, reset the step size to $h = 1$ (see the first note in the Comment below).

(a) Use Euler's method to estimate to three decimal places the value of $y(6)$, where $y(x)$ satisfies the initial-value problem

$$\frac{dy}{dx} = x - y - 3, \quad y(1) = 1.$$

(For the step size h, use in turn the values 1, 0.5, 0.1, 0.01, 0.001 and 0.0001.)

(b) Check that the function $y = x - 4 + 4e^{1-x}$ is the solution to the initial-value problem in part (a). Hence find $y(6)$ exactly. Does this agree with the value that you obtained using Euler's method in part (a)?

Solutions are given on page 34.

Comment

◇ If you do not start by setting $h = 1$, then any other change will cause recalculation for the most recently-used value of the step size, $h = 0.0001$, which may be time-consuming. Alternatively, recall that any Mathcad calculation can be interrupted by pressing [Esc] and then clicking 'OK' in the resulting option box. You may prefer to change here to 'manual calculation mode'.

See the Comment for Activity 4.4 of Chapter B1, in Computer Book B.

◇ There is no need to alter any part of the file before page 3, nor to change the parameters which define the grid for the direction field. On page 3 you need to alter the direction field function definition to $f(x, y) = x - y - 3$. Also, the values of x_0 and y_0 should both be set to 1, and the value of *xval* to 6.

◇ Using the suggested values of h in turn, the numerical estimates y_N appear to converge and the approximate solution curves do likewise. The 'chain of line segments' is visible for $h = 1$ and for $h = 0.5$, but no departures from smoothness are apparent on the graph for smaller step sizes.

Now close Mathcad file 121C3-02.

Solutions to Activities

Chapter C1

Solution 5.1

The derivative of $x^2/5$ is -0.4 when $x = -1$, and 0.2 when $x = 0.5$.

The function $f(x) = x^3 - x^2 - 3x + 4$ has derivative equal to zero at approximately $x = -0.72$ and $x = 1.39$

Solution 5.2

Where more than one expression is given below for a solution, the first is similar to the Mathcad output and the second is a form that you are more likely to obtain by hand. (You found each of the derivatives in parts (b)–(h) by hand in the main text of Chapter C1, as indicated by the references below.)

(a) $3x^2 - 2x - 3$

(b) $4\pi r^2$

See Activity 3.4(a).

(c) $$\frac{(\cos(t) - 2t)}{\exp(t)} - \frac{(\sin(t) - t^2)}{\exp(t)}$$
$$= \frac{\cos t - \sin t + t^2 - 2t}{e^t}$$

See Exercise 4.2(b).

(d) $2\cos(x^2)x = 2x\cos(x^2)$

See pages 48–49

(e) $-4\sin(4x)$

See Activity 4.7(a).

(f) $$2t\ln(t) + \frac{(t^2 + 3)}{t}$$

See Activity 4.2(c).

(g) $$\frac{1}{u(u^2 + 3)} - 2\frac{\ln(u)}{(u^2 + 3)^2}u = \frac{u^2 + 3 - 2u^2 \ln u}{u(u^2 + 3)^2}$$

See Activity 4.4(b)

(h) $$\frac{(\exp(t) + 1)}{\exp(t) + t} = \frac{e^t + 1}{e^t + t}$$

See Exercise 4.3(d).

Solution 5.3

Step 1: The stationary points of the function

$$f(x) = 0.125\sqrt{1 + x^2} + 0.0625(2 - x)$$

are at $x = \pm 0.577$ (to 3 d.p.).

Step 2: Only the positive stationary point, 0.577, lies inside the interval $[0, 2]$. The values of f at the interval endpoints are

$$f(0) = 0.250 \quad \text{and} \quad f(2) = 0.280,$$

while the value of f at the stationary point in the interval is

$$f(0.577) = 0.233 \quad \text{(all to 3 d.p.)}.$$

Step 3: Hence the minimum value of $f(x)$ for x in the interval $[0, 2]$ is 0.233 at $x = 0.577$ (both to 3 d.p.).

Hence the solution to the orienteer's problem is to aim to join the path at a distance 0.577 km from the fixed point O on the path.

(This agrees with the answer 0.58 km found in Activity 5.7(a) of Chapter A3 in Computer Book A.)

Solution 5.4

(c) *Step 1:* The stationary points of the function

$$f(v) = \frac{v}{5 + v + 0.02v^2}$$

are at $v = \pm 15.811$ (to 3 d.p.).

Step 2: Only the positive stationary point, 15.811, lies inside the interval $[0, 35]$. The values of f at the interval endpoints are

$$f(0) = 0 \quad \text{and} \quad f(35) = 0.543,$$

while the value of f at the stationary point in the interval is

$$f(15.811) = 0.613 \text{ (all to 3 d.p.)}.$$

Step 3: Hence the maximum value of $f(v)$ for v in the interval $[0, 35]$ is 0.613 at $v = 15.811$ (both to 3 d.p.).

(d) According to the model, a maximum traffic flow rate of about 0.61 vehicles per second can be achieved, by keeping the speed of traffic at about 16 $\mathrm{m\,s^{-1}}$ (that is, about 57 km per hour or 35 mph).

Chapter C2

Solution 5.2

In each case, c is an arbitrary constant which has been added to the expression given by Mathcad. The answers to parts (a) and (b) agree with those obtained earlier by hand.

(a) $\displaystyle\int \left(\frac{1}{x} + e^{3x}\right) dx = \ln(x) + \tfrac{1}{3}\exp(3x) + c$

See Activity 1 2(a)

(b) $\displaystyle\int \left(\frac{3}{y^4} + 5\sin(5y)\right) dy = -\frac{1}{y^3} - \cos(5y) + c$

See Exercise 1.1(b).

(c) $\displaystyle\int (a + \cos(ax))\, dx = ax + \frac{\sin(ax)}{a} + c$

Solution 5.3

In each case, c is an arbitrary constant which has been added to the expression given by Mathcad. The Mathcad answers in parts (a), (b) and (d) resemble closely those obtained by hand in the main text.

(a) $\displaystyle\int (x-3)(x-1)\, dx = \tfrac{1}{3}x^3 - 2x^2 + 3x + c$

See Example 2.1(a).

(b) $\displaystyle\int \frac{2x-3}{\sqrt{x}}\, dx = \tfrac{4}{3}x^{3/2} - 6\sqrt{x} + c$

See Example 2.1(c).

(c) $\displaystyle\int \sin^2 x\, dx = -\tfrac{1}{2}\cos(x)\sin(x) + \tfrac{1}{2}x + c$

In Activity 2.3(a), the answer $\frac{1}{2}x - \frac{1}{4}\sin(2x) + c$ was obtained by hand. The equivalence of this and the Mathcad expression follows from the double-angle formula $\sin(2x) = 2\sin x\cos x$.

(d) $\displaystyle\int \frac{x}{x^2+1}\, dx = \tfrac{1}{2}\ln(x^2+1) + c$

See Activity 2.5(b)(iii).

(e) $\displaystyle\int ue^{3u}\, du = \tfrac{1}{3}u\exp(3u) - \tfrac{1}{9}\exp(3u) + c$

(f) $\displaystyle\int x^2\ln(5x)\, dx = \tfrac{1}{3}x^3\ln(5x) - \tfrac{1}{9}x^3 + c$

(g) $\displaystyle\int \frac{1}{\sqrt{9-t^2}}\, dt = \arcsin(\tfrac{1}{3}t) + c$

(The function arcsin is represented in Mathcad by asin.)

Solution 5.5

The answers are presented in a form as close as possible to the Mathcad output (which by default is given to 3 decimal places). Each answer agrees with that obtained by hand in the main text.

(a) $\displaystyle\int_0^2 e^t\, dt = \exp(2) - 1 = 6.389$

See Activity 4.4(b).

(b) $\displaystyle\int_0^{\pi/4} (\cos(5x) + 2\sin(5x))\, dx = \tfrac{1}{10}\sqrt{2} + \tfrac{2}{5}$
$= 0.541$

See Exercise 4.1(a).

(c) $\displaystyle\int_1^2 \frac{6}{u^2}\, du = 3$

See Exercise 4.1(b). (This answer is the Mathcad outcome from symbolic evaluation.)

(d) $\displaystyle\int_0^{\pi} e^t \sin t\, dt = \tfrac{1}{2}\exp(\pi) + \tfrac{1}{2} = 12.070$

See Exercise 4.1(c).

Solution 5.7

The answers are given by Mathcad to 3 decimal places.

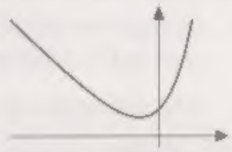
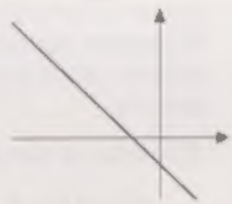
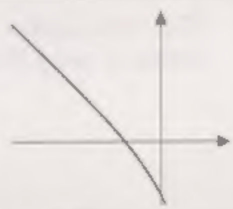
(a) $\displaystyle\int_0^1 t^2\, dt = 0.333$

(b) $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\, dx = 1.571$

Chapter C3

Solution 4.1

(a) The solution curves are described and sketched in the following table.

(x_0, y_0)	Description	Sketch
$(0, 0)$	Rough U-shape; steeper to right of minimum than to left.	
$(0, 1)$	Similar, but with minimum to the left and higher.	
$(0, 2)$	Similar, but with minimum still further to the left and higher.	
$(0, -1)$	Straight line $y = -x - 1$.	
$(0, -2)$	Downward curve with gradient decreasing.	

(b) The solution curves show three distinct types of behaviour:

(i) Any solution curve through a point above the line $y = -1 - x$ remains above that line. It decreases to a minimum and then increases.

(ii) The line $y = -1 - x$ is itself the solution curve through any point on that line.

(iii) Any solution curve through a point below the line $y = -1 - x$ remains below that line and decreases.

The solution curves through the points $(-3, -1)$, $(-1, 0)$ and $(4, 2)$ are of types (iii), (ii) and (i), respectively.

Solution 4.2

(a) (ii) The slope of the direction field at a point (x_0, y_0) appears to depend on the choice of x_0 alone and not on that of y_0. Correspondingly, there will be solution curves of just one type, with any two such curves differing only by a vertical translation. (This is a consequence of the fact that the function $f(x, y)$ in this case depends only on x.)

(iii) All choices for x_0 and y_0 give a solution curve which increases and decreases alternately, but has a rising trend. Each such curve may be obtained from that shown below by a vertical translation.

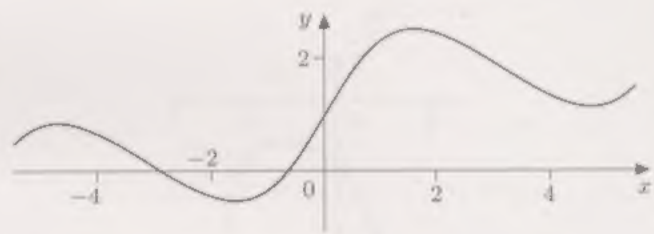


Figure S3.1

(b) (ii) The effect of the direction field on solution curves in this case is not so clear-cut. However, it appears that there may be three types of solution curve, as detailed below.

(iii) There are solution curves that cross the x-axis and are increasing, as in the graph below. If $x_0 = 0$, then such curves are obtained by a choice of y_0 such that $-1.25 \leq y_0 \leq 1.25$.

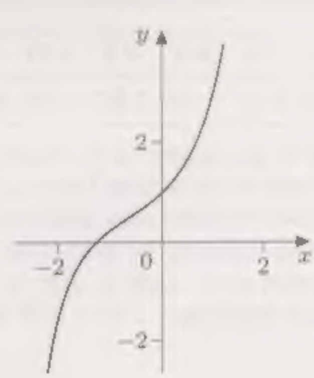


Figure S3.2

There are solution curves that lie above the x-axis, each having a minumum in the second quadrant, as in the graph in Figure S3.4. If $x_0 = 0$, then such curves are obtained by a choice of y_0 such that $y_0 \geq 1.26$.

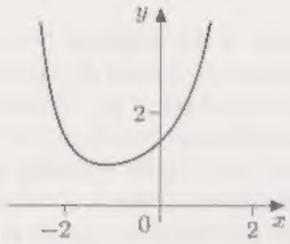


Figure S3.3

There are solution curves that lie below the x-axis, each having a maximum in the fourth quadrant, as in the graph below. If $x_0 = 0$, then such curves are obtained by a choice of y_0 such that $y_0 \leq -1.26$.

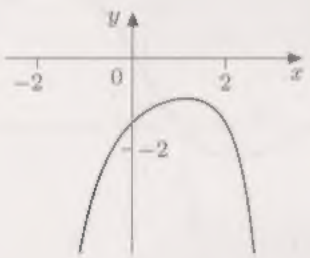


Figure S3.4

Solution 4.4

(a) The graph extends to $x = 9$, but the portion to the left of $x = 5$ remains unchanged. In particular, the step size h does not alter, though the number N of steps does (so that the graph reaches $x = 9$ rather than $x = 5$).

(b) A table of the values obtained is below.

h	1	0.5	0.2	0.1	0.01	0.001	0.0001
y_N	3.916	3.347	3.002	2.887	2.783	2.772	2.771

The values of y_N appear to be converging and, given the level of agreement between the last two estimates, it seems reasonable to deduce from them an estimate for $y(9)$ that is accurate to one decimal place, that is, $y(9) = 2.8$. (In fact, it looks likely that $y(9) = 2.77$ to 2 decimal places.)

Solution 4.5

(a) The table below gives the values obtained using Euler's method, from file 121C3-02.

h	1	0.5	0.1	0.01	0.001	0.0001
y_N	2.000	2.004	2.021	2.026	2.027	2.027

The agreement of the last two estimates suggests that $y(6) = 2.027$ to three decimal places.

(b) For the given function $y = x - 4 + 4e^{1-x}$, we have

$$y(1) = 1 - 4 + 4e^0 = 1,$$

so that the initial condition of the problem in part (a) is satisfied.

The derivative of the given function is

$$\frac{dy}{dx} = 1 - 4e^{1-x},$$

whereas we have

$$\begin{aligned} x - y - 3 &= x - (x - 4 + 4e^{1-x}) - 3 \\ &= 1 - 4e^{1-x}. \end{aligned}$$

Hence the given function also satisfies the differential equation from part (a), and therefore satisfies all the conditions of the initial-value problem.

The value of this solution function at $x = 6$ is

$$y(6) = 6 - 4 + 4e^{1-6} = 2 + 4e^{-5} \simeq 2.027.$$

This value agrees to 3 decimal places with that obtained using Euler's method.

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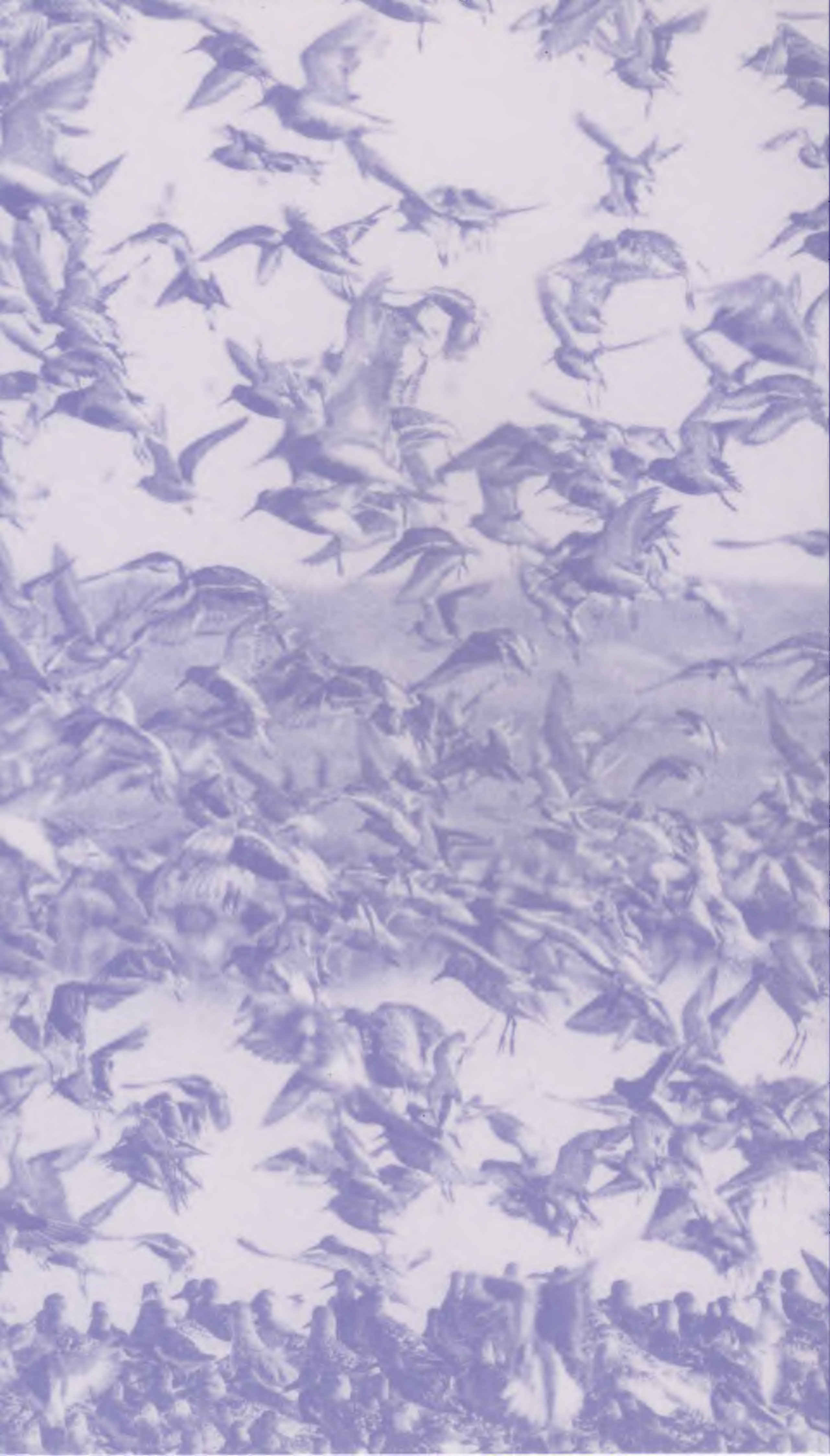

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